Uniformly Most Powerful Tests (UMPT)

Ist case: Ho is simple and H_1 is Composite

(i) One sided Composite H_1

Suppose in a normal distn, we want to test $H_0: \mu = \mu_0$ Vs $H_1: \mu > \mu_0$. Here H_0 is simple but H_1 is one sided Composite. Also μ_0 is specified. Now if $X \sim N(\mu, 1)$, then the BCR for the test of size $\alpha = .05$ is

$$\omega = \left\{ \underline{x} : \bar{x} \geq \underline{\underline{\mu_0}} + \frac{1.645}{\sqrt{n}} \right\} \quad \left(\begin{array}{l} H_0: \mu = \mu_0 \\ H_1: \mu = \mu_1 > \mu_0 \end{array} \right.$$

where $\bar{x}$ is the mean of random sample from $N(\mu, 1)$. As this BCR 'ω' depends on μ_0 (under H_0), but not on μ_1 (under H_1), provided that $\mu_1 > \mu_0$. We would, therefore, be leading to the same BCR in the cases

$H_0: \mu = \mu_0$ Vs $H_1: \mu = \mu_1 = \mu_0 + a$, $a > 0$ (as $\mu_1 > \mu_0$).

The test based on the CR 'ω' above is most powerful test of size $\alpha = .05$ for all simple alternatives of the form $H_1: \mu = \mu_1 > \mu_0$. Thus, we say that test based on ω is UMPT of $H_0: \mu = \mu_0$ Vs $H_1: \mu > \mu_0$.

This UMPT is provided by NPL by modifying the above H_1 as $H_1: \mu = \mu_1 (> \mu_0)$, as

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NPL is used only for testing simple H_0 versus simple H_1. Here H_0 is simple and H_1 has been modified to simple case. (by putting $\mu = \mu_1 > \mu_0$).

Note: Later on, we shall see that UMPT does not exist for two sided case, as we cannot define a unique BCR for each H_1 (i.e. $H_1: \mu \neq \mu_0$).

(For example $H_1: \mu \neq \mu_0 \Rightarrow H_1: \mu > \mu_0$ or $\mu < \mu_0$

Now for $H_1: \mu > \mu_0$ or $\mu < \mu_0$ $\rightarrow$ one gets different BCR's.).

Definition: The CR '$\mathcal{C}$' is UMPCR of size α for testing simple H_0 Vs a one sided composite H_1 if the set (CR) $\mathcal{C}$ is the BCR of size α for testing H_0 against each simple H_1 (in composite H_1).

A test based on this UMPCR, $\mathcal{C}$, is called UMP Test of size α for testing simple H_0 VS Composite H_1.

Exp let $\underline{x}$ be a r. sample from $N(0,\theta)$.
Show that there exists a UMPT for testing $H_0: \theta = \theta_0$ Vs $H_1: \theta > \theta_0$.

The pdf of $N(0,\theta)$ is given by

$$f(x,\theta) = \theta^{-1/2} (2\pi)^{-1/2} e^{-\frac{1}{2\theta} x^2}, \quad -\infty < x < \infty$$

The LF is

$$f_{\underline{x}}(\underline{x}) = (2\pi)^{-n/2} \theta^{-n/2} e^{-\Sigma x^2/2\theta^2}$$

Under H_0, the L.F is

$$f_0(\underline{x}) = (2\pi)^{-n/2} \theta_0^{-n/2} e^{-\frac{1}{2\theta_0}\Sigma x^2} \quad \text{—} (1)$$

Under $H_1: \theta = \theta_1 (>\theta_0)$ ($\rightarrow$ adjusted H_1)

the L.F is

$$f_1(\underline{x}) = (2\pi)^{-n/2} \theta_1^{-n/2} e^{-\frac{1}{2\theta_1}\Sigma x^2}$$

Using NPL, we have

$$\frac{f_1(\underline{x})}{f_0(\underline{x})} \geq k, \text{ for each } \underline{x} \in C$$

$$\frac{\cancel{(2\pi)^{-n/2}} \theta_1^{-n/2} e^{-\frac{1}{2\theta_1}\Sigma x^2}}{\cancel{(2\pi)^{-n/2}} \theta_0^{-n/2} e^{-\frac{1}{2\theta_0}\Sigma x^2}} \geq k, \quad " \quad " \quad "$$

$$\left(\frac{\theta_1}{\theta_0}\right)^{-n/2} \exp\left[-\frac{\Sigma x^2}{2\theta_1} + \frac{\Sigma x^2}{2\theta_0}\right] \geq k, \quad " \quad " \quad "$$

$$\exp\left[\frac{\Sigma x^2}{2}\left\{\frac{1}{\theta_0} - \frac{1}{\theta_1}\right\}\right] \geq k_1, \quad " \quad " \quad "$$

Taking log on both sides to have

$$\frac{\Sigma x^2}{2}\left[\frac{\theta_1 - \theta_0}{2\theta_0}\right] \geq k_2, \quad " \quad " \quad "$$

$$\Sigma x^2 (\theta_1 - \theta_0) \geq k_3, \quad " \quad " \quad "$$

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$\Sigma x^2 \geq c$ $(\because \theta_1 > \theta_0)$ for each $x \in \mathscr{C}$

Thus, the BCR for testing

$H_0 : \theta = \theta_0$ Vs $H_1 : \theta = \theta_1 (> \theta_0)$

is $\mathscr{C} = \{ \underline{x} : \Sigma x^2 \geq c \}$; where c is to be determined so that the test is of size α i.e. $P[\underline{x} \in \mathscr{C} / H_0] = \alpha$

$\Rightarrow P[\Sigma x^2 \geq c / H_0 : \theta = \theta_0] = \alpha$

Thus, the Constant 'c' in the BCR depends on θ under H_0 (i.e θ_0) and is independent of $\theta (= \theta_1)$ under H_1, as long as $\theta_1 > \theta_0$.

Hence, for each simple H_1 of the form $H_1 : \theta = \theta_1 (> \theta_0)$, this critical region ~~BCR~~ remains 'BCR', that is, $\mathscr{C}$ is the BCR for $H_0 : \theta = \theta_0$ Vs $H_1 : \theta > \theta_0$. Thus, the BCR is uniformly best or most powerful critical region (UMPCR) and the test based on this UMPCR is UMP Test of size α.

Under H_0, $\Sigma x^2 / \theta_0^2 \sim \chi^2_{(n)}$, so

$P[\Sigma x^2 / \theta_0 \geq c / \theta_0] = \alpha$

$\Rightarrow P[\chi^2_{(n)} \geq c / \theta_0] = \alpha$ ——— (2)

If α and n is known, c can be determined.

Let $n = 10$, $\alpha = .05$, then

$P[\chi^2_{(10)} \geq 18.3] = 0.05$ ——— (3)

from (2) & (3) $c / \theta_0 = 18.3 \Rightarrow c = 18.3\, \theta_0$.

Exp Consider two independent normal populations $N(\mu_1, 400)$ and $N(\mu_2, 225)$. let $\theta = \mu_1 - \mu_2$. suppose $\bar{x}$ and $\bar{y}$ denote the means of observed samples of size n each from the populations respectively. Consider the hypothesis

$H_0: \theta = 0$ Vs $H_1: \theta > 0$

obtain UMPCR. Find n and C such that

$\pi(\theta_0) = \pi(0) = 0.10$, $\pi(\theta_1) = \pi(10) = 0.90$

Solⁿ: $x_i \sim N(\mu_1, 400)$, $y_i \sim N(\mu_2, 225)$, $i = 1, 2, \ldots, n$.

let $H_0: \theta = 0$ Vs $H_1: \theta = \theta_1 (> 0)$

let $z_i = x_i - y_i \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

$z_i \sim N(\theta, 625)$, $\because \theta = \mu_1 - \mu_2$, $\sigma_1^2 + \sigma_2^2 = 225 + 400 = 625$

$H_0: \theta = 0$ Vs $H_1: \theta = \theta_1 (> 0)$

$$f(z_i) = (625)^{-1/2} (2\pi)^{-1/2} e^{-\frac{1}{2\sigma^2}(z_i - \theta)^2}$$

The L.F is

$$f(\underline{z}) = (625)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2(625)}\Sigma(z - \theta)^2}$$

Under $H_0: \theta = 0$, the L.F is

$$f_0(\underline{z}) = (625)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2(625)}\Sigma(z - 0)^2}$$

$$= (625)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{1250}\Sigma z^2} \quad \text{--- (1)}$$

Under $H_1: \theta = \theta_1$, the L.F is

$$f_0(\underline{z}) = (625)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{2(625)}\Sigma(z - \theta_1)^2} \quad \text{--- (2)}$$

Now by NPL, we have

$$\frac{f_1(\underline{z})}{f_0(\underline{z})} \geq k, \text{ for each } \underline{z} \in \omega$$

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$$\frac{e^{-\frac{1}{1250}\sum(z_i-\theta_1)^2}}{e^{-\frac{1}{250}\sum z_i^2}} \geq k, \quad '' \quad '' \quad ''$$

$$\Rightarrow \quad e^{-\frac{1}{1250}\sum(z-\theta)^2+\frac{1}{1250}\sum z^2} \geq k, \quad '' \quad ''$$

Taking log on both sides to have

$$-\frac{1}{1250}\sum(z-\theta_1)^2+\frac{1}{1250}\sum z^2 \geq k_1, \quad '' \quad '' \quad ''$$

$$-\frac{1}{1250}\left[\cancel{\sum z^2}+n\theta_1^2-2\theta_1\sum z-\cancel{\sum z^2}\right] \geq k_1, \quad '' \quad '' \quad ''$$

$$n\theta_1 - 2\theta_1\sum z \leq k_2, \quad '' \quad '' \quad ''$$

$$-2\theta\sum z \leq k_3, \quad '' \quad '' \quad ''$$

$$\sum z_i \geq k_4, \quad '' \quad '' \quad ''$$

$$\bar{z} \geq c, \quad '' \quad '' \quad '' \quad \left(\bar{z}=\frac{\sum z}{n}\right)$$

$$\Rightarrow \quad \frac{1}{n}\sum(x_i-y_i) \geq c, \quad '' \quad '' \quad ''$$

$$\Rightarrow \quad \bar{x}-\bar{y} \geq c, \quad '' \quad '' \quad ''$$

Thus, the BCR is $\omega=\{(\underline{x},\underline{y}): \bar{z}=\bar{x}-\bar{y} \geq c\}$ where c satisfies $P[\bar{z}=\bar{x}-\bar{y} \geq c \,/ H_0: \theta=0]=\alpha$. Since this c (or ω) is to be determined so that the test is of size α (under H_0), therefore, this ω is independent of H_1 (whatever θ under H_1 may be). Thus, for each H_1 (simple), this BCR remains BCR and hence is UMPCR.

Now as $P[\bar{x}-\bar{y} \geq c \,/ H_0] = 0.10$

$$\Rightarrow \quad P\left[\frac{\bar{x}-\bar{y}}{\sqrt{625/n}} \geq \frac{c-0}{\sqrt{625/n}}\right] \text{ —— } (3)$$

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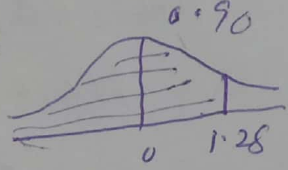
Since $\bar{x}-\bar{y} \sim N(0, \frac{625}{n})$ under H_0.

$\therefore$ $Z = \dfrac{\bar{x}-\bar{y}}{\sqrt{625/n}} \sim N(0,1)$ under H_0.

from (3) $P[Z \geq \frac{\sqrt{n}\,c}{25}] = 0.90$ —— (4)

Also we know that from standard normal distribution:

$P[Z \geq 1.28] = 0.90$ —— (5)



from (4) & (5) $\dfrac{\sqrt{n}\,c}{25} = 1.28$ —— (A)

Again $P[\bar{x}-\bar{y} \geq c \,/ H_1 : \theta = 10] = 0.90$

$\Rightarrow P\left[\dfrac{(\bar{x}-\bar{y})-10}{\sqrt{625/n}} \geq \dfrac{c-10}{\sqrt{625/n}}\right] = 0.90$

$P\left[Z \geq \dfrac{\sqrt{n}\,(c-10)}{25}\right] = 0.90$

$P\left[Z \leq \dfrac{\sqrt{n}\,(c-10)}{25}\right] = 0.10$ —— (6)

Also from standard normal distribution

$P[Z \leq -1.28] = 0.10$ —— (7)

from (6) and (7) $\dfrac{\sqrt{n}\,(c-10)}{25} = -1.28$ —— (B)

solving (A) and (B) for n and c, we get

$n = 40.96 \simeq 41, \quad c \doteq 5.$

Note: $\bar{x}-\bar{y} \sim N(10, \frac{625}{n})$ under H_1.

$Z = \dfrac{\bar{x}-\bar{y}}{\sqrt{625/n}} \sim N(0,1)$.

Exp (Practical): let $\underline{x}$ be a random sample (45)
of size n from $N(\theta, 16)$. Find the value of n and c
and UMPT of $H_0: \theta = 25$ Vs $H_1: \theta \leq 25$ when
$\pi(25) = 0.10$, $\pi(23) = 0.90$.
Draw the graph of $\pi(\theta)$, for $\theta = 25, 24, 23, \ldots, 20$.

The pdf of $N(\theta, 16)$ is

$$f(x,\theta) = (16)^{-1/2} (2\pi)^{-1/2} e^{-\frac{1}{2(16)}(x-\theta)^2}, \quad -\infty < x < \infty$$

The L.F is

$$f(\underline{x}) = (16)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{32}\Sigma(x-\theta)^2}$$

The L.F under $H_0: \theta = 25$ is

$$f_0(\underline{x}) = (16)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{32}\Sigma(x-25)^2}$$

The L.F under $H_1: \theta = \theta_1 (<25)$ is

$$f_1(\underline{x}) = (16)^{-n/2} (2\pi)^{-n/2} e^{-\frac{1}{32}\Sigma(x-\theta_1)^2}$$

By NPL, we have

$$\frac{f_1(\underline{x})}{f_0(\underline{x})} \geq k, \text{ for each } \underline{x} \in C$$

$$\Rightarrow \frac{e^{-\frac{1}{32}\Sigma(x-\theta_1)^2}}{e^{-\frac{1}{32}\Sigma(x-25)^2}} \geq k, \text{ for each } \underline{x} \in C$$

$$\Rightarrow e^{-\frac{1}{32}\left[\Sigma(x-\theta_1)^2 - \Sigma(x-25)^2\right]} \geq k, \quad " \quad " \quad "$$

Taking natural log on both sides.

$$-\frac{1}{32}\left[\Sigma(x-\theta_1)^2 - \Sigma(x-25)^2\right] \geq k_1, \quad " \quad " \quad "$$

$$\Rightarrow \Sigma(x-\theta_1)^2 - \Sigma(x-25)^2 \leq k_2, \quad " \quad " \quad "$$

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(46)

$\Rightarrow \sum x^2 + n\theta_1^2 - 2\theta_1 \sum x - \sum x^2 - n(25)^2 + 2(25)\sum x \not< k_2$, for each $\underline{x} \in \omega$

$\Rightarrow \sum x[50 - 2\theta_1] \not< k_3$, " " "

$\sum x \not\geq c$ $\because \theta_1 < 25$

$\because 2\theta_1 < 50$

Thus, the BCR for $H_0: \theta = 25$ Vs $H_1: \theta = \theta_1 (<25)$ is $\omega = \{ \underline{x} : \sum x \leq c \}$, where c satisfies

$$P[\underline{x} \in \omega / H_0] = \alpha \Rightarrow P[\sum x \leq c / H_0 : \theta = 25] = \alpha$$

Since the BCR ω (and also c) does not depend on $\theta (= \theta_1)$ under H_1 (c is determined under $H_0: \theta = 25$), therefore this BCR remains the same for testing $H_0: \theta = 25$ versus all simple alternatives (of the form $H_1: \theta = \theta_1 (<25)$) in Composite $H_1: \theta < 25$.

Thus, the BCR is UMPCR of size α and hence the test is UMPTest of size α.

Now as

$$\pi(\theta) = P[\text{Reject } H_0] = P[\sum x \leq c]$$

$$\pi(\theta_0) = \pi(25) = P[\sum x \leq c / H_0 : \theta = 25] = 0.10$$

$$\Rightarrow P[\bar{x} \leq c/n / H_0 : \theta = 25] = 0.10 \quad \text{---} (*)$$

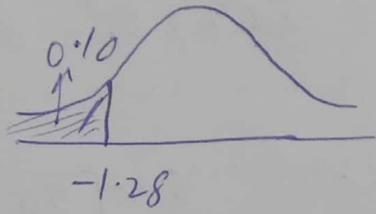
But as $\bar{x} \sim N(\theta = 25, 16)$ under H_0,

or $z = \dfrac{\bar{x} - 25}{\sqrt{16/n}} \sim N(0,1)$ under H_0,

Thus, $(*) \Rightarrow P\left[z \leq \dfrac{c/n - 25}{\sqrt{16/n}} \right] = 0.10$ --- (1)

We also know that

$P[Z \le -1.282] = 0.10$ ——— (2)



from (1) + (2) $\dfrac{\frac{c}{n} - 25}{\sqrt{16/n}} = -1.282$ ——— (A)

Again $\pi(\theta_1) = \pi(23) = P[\Sigma x \le c / H_1 : \theta = 23] = 0.90$

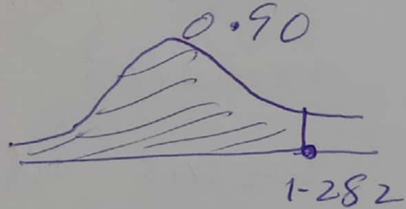
$\Rightarrow P[\bar{x} \le c/n \, / H_1 : \theta = 23] = 0.90$

$\Rightarrow P\left[\dfrac{\bar{x} - 23}{\sqrt{16/n}} \le \dfrac{c/n - 23}{\sqrt{16/n}}\right] = 0.90$

$\Rightarrow P\left[Z \le \dfrac{\left(\frac{c}{n} - 23\right)}{\sqrt{16/n}}\right] = 0.90$ ——— (3)

We also know that

$P[Z \le 1.282] = 0.90$ ——— (4)



from (3) + (4) :

$\dfrac{\left(\frac{c}{n} - 23\right)}{\sqrt{16/n}} = 1.282$ ——— (B)

we

Solving (A) + (B) for n and c, we get $n = 26.21 \approx 27$

$c \doteqdot \boxed{}$.

<u>Graph</u>: $\pi(\theta) = P[\Sigma x \le c / H_1 : \theta = 25, 24, \ldots, 20]$

calculate $\pi(\theta)$ for $\theta = 25, \ldots, 20$ and then plot $\pi(\theta)$ versus θ.

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Most Powerful Function

Let $\underline{x} \sim N(\mu, \sigma_0^2)$. In the context of simple H_1, we observed that the performance of test was measured by its power. that is, $\pi(\theta_1) = P[\underline{x} \in \mathcal{C} / H_1]$.

This ($\pi(\theta_1)$) should be maximum so that the test is of size α. The CR $\mathcal{C}$ is called BCR.

Similarly, for the composite alternative $H_1: \mu > \mu_0$ (where μ has more than one value), the power will change as μ changes within H_1 ($\pi(\mu) = P[\underline{x} \in \mathcal{C} / H_1 : \mu > \mu_0]$).

Thus, we are dealing with the power function $\pi(\mu)$ given by

$$\pi(\mu) = P[\underline{x} \in \mathcal{C} / H_1 : \mu > \mu_0]$$

$$= P\left[\bar{x} > \mu_0 + \frac{1.645\,\sigma_0}{\sqrt{n}} \Big/ \mu > \mu_0\right]$$

$$= P\left[\frac{\bar{x} - \mu}{\sigma_0/\sqrt{n}} > \frac{\mu_0 + \frac{1.645\,\sigma_0}{\sqrt{n}} - \mu}{\sigma_0/\sqrt{n}}\right]$$

First find c $P[\bar{x} \geq c / H_0]$ under H_0.

$$= P\left[Z > 1.645 - \frac{\sqrt{n}(\mu - \mu_0)}{\sigma_0}\right]$$

For particular value of μ, we can evaluate $\pi(\mu)$ using the table of standard normal distribution. For instance, if $\mu = \mu_0$,

$$\pi(\mu_0) = P[Z > 1.645] = 0.05 = \alpha.$$

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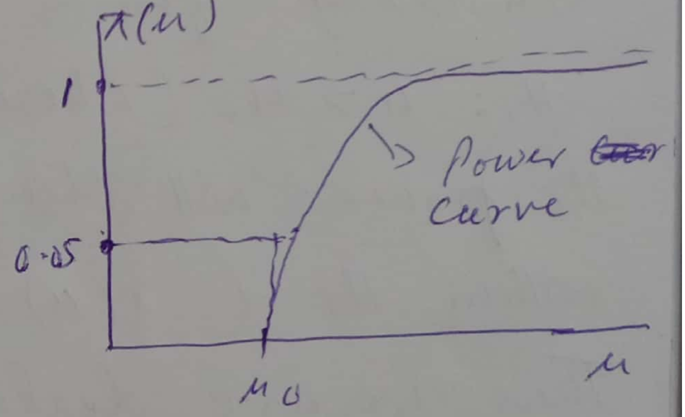
If $\mu \to \infty$, with n and μ_0 fixed,

$$\pi(\mu) \longrightarrow \lim_{\mu \to \infty} P\left[Z > 1.645 - \frac{(\mu - \mu_0)\sqrt{n}}{\sigma_0}\right]$$

$$\pi(\mu) \longrightarrow 1$$

that is, probability of rejecting H_0 when H_1 is true, approaches 1 (maximum power).

that is, the power of test is increasing, for fixed n and μ_0, for increasing value of μ. Thus, the Test becomes more and more powerful as $\mu \to \infty$.



The Complement of power curve is

$$O \cdot C = 1 - \pi(\mu) = 1 - P[\underline{x} \in \omega_c / H_1]$$

Thus, the O.C decreases for increasing value of μ (ie β decreases as $\mu \to \infty$).

BCR AND SUFFICIENT STATISTIC

It can easily be verified that in testing a simple H_0 vs a simple H_1, the BCR is necessarily a function of sufficient statistic for θ, if one exists. We may note from Neyman factorization criterion that

$$f_{\underline{x}}(\underline{x};\theta) = g(t:\theta)\,h(\underline{x})$$

Where T is sufficient statistic for θ. Then testing $H_0: \theta = \theta_0$ Vs $H_1: \theta = \theta_1$ the NPL gives

$$\frac{f_1(\underline{x})}{f_0(\underline{x})} \geq k \Rightarrow \frac{g(t:\theta_1)}{g(t:\theta_0)} \geq k, \text{ for } \underline{x} \in \omega$$

This can be solved interms of T, to get BCR. Thus, the BCR will be a function of sufficient statistic T, if it exists.

UMPCR (UMPT) AND SUFFICIENT STATISTIC

In testing a simple H_0 Vs a simple H_1, it follows that if a BCR exists, then it is a function of sufficient statistic.

Again, it follows that if a common BCR (UMPCR) exists (for testing simple H_0 Vs one sided H_1), providing us with a a UMP test of size α against H_1 and if T is sufficient statistic for the parameter(s), then the UMPCR will be a function of this T, provided that there is a single sufficient statistic for θ.

Note that there will always a UMP test if

$H_0: \theta = \theta_0$ Vs $H_1: \theta < \theta_0$ or

$H_0: \theta = \theta_0$ Vs $H_1: \theta > \theta_0$, for $0 < \alpha < 1$.

There is, however, in general no UMP Test of $H_0: \theta = \theta_0$ Vs $H_1: \theta \neq \theta_0$ even when there is a sufficient statistic.

Exp: $X_1, X_2 ..., X_n \sim U(-\theta, \theta)$.

There is a joint sufficient statistic (Y_1, Y_n) for θ (no single sufficient statistic exists).

Here UMPCR may be a function of either Y_1 or Y_n (but not both Y_1, Y_n).

Points i) The power, $\pi(\theta)$, is a measure of sensitiveness of the test. Therefore, if other things are identical (i.e size of test etc), the Comparison of tests (efficiency) is done through their respective powers.

ii) If $\pi_1(\theta)$ and $\pi_2(\theta)$ are the tests corresponding to two tests k_1 and k_2. Then the test with k_1 is more efficient if $\pi_1(\theta) > \pi_2(\theta)$.

iii) The Condition that 'α' is kept constant and attempt is made to minimize β, is the basic concept of NPL.

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Problem: let $\underline{x}$ be a random sample from
$$f(x,\theta) = \frac{\theta^{q} x^{q-1} e^{-\theta x}}{\sqrt{q}}, \quad x > 0, \quad q \text{ is known.}$$
Find MP Test of size α for $H_0: \theta = \theta_0$ Vs $H_1: \theta = \theta_1$ $(\theta_1 > \theta_0)$ and show that it is also UMP test for $H_0: \theta = \theta_0$ Vs $H_1: \theta > \theta_0$ (Composite). In case, when $q = \frac{1}{n}$, show that the Power function of this test is $1-(1-\alpha)^{\theta/\theta_0}$.

Sol^n:
$$f_x(x,\theta) = \frac{\theta^{q} x^{q-1} e^{-\theta x}}{\sqrt{q}}, \quad x > 0$$

The LF is
$$f(\underline{x},\theta) = \frac{\theta^{nq} (\Pi x_i)^{q-1} e^{-\theta \Sigma x}}{(\sqrt{q})^n}$$

Under H_0 and H_1, the L.F's are
$$f_0(\underline{x}) = \frac{\theta_0^{nq} (\Pi x_i)^{q-1} e^{-\theta_0 \Sigma x}}{(\sqrt{q})^n}$$
and
$$f_1(\underline{x}) = \frac{\theta_1^{nq} (\Pi x_i)^{q-1} e^{-\theta_1 \Sigma x}}{(\sqrt{q})^n}$$

Using NPL, we have
$$\frac{f_1(\underline{x})}{f_0(\underline{x})} \geq k \Rightarrow \frac{\theta_1^{nq} \cancel{(\Pi x_i)^{q-1}} e^{-\theta_1 \Sigma x} / \cancel{(\sqrt{q})^n}}{\theta_0^{nq} \cancel{(\Pi x_i)^{q-1}} e^{-\theta_0 \Sigma x} / \cancel{(\sqrt{q})^n}} \geq k$$

$$\Rightarrow \left(\frac{\theta_1}{\theta_0}\right)^{nq} e^{-\theta_1 \Sigma x + \theta_0 \Sigma x} \geq k, \text{ for } \underline{x} \in \omega$$

$$\Rightarrow e^{-\Sigma x(\theta_1 - \theta_0)} \geq k_1, \quad \text{"} \quad \text{"}$$

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Taking log on both sides:

$$-\Sigma x(\theta_1-\theta_0) \geq k_2 \quad , \text{ for } \underline{x} \in \mathscr{C}$$

$$\Rightarrow \quad \Sigma x \leq c \qquad \because \theta_1 > \theta_0$$

Thus, the BCR is $\mathscr{C} = \{ \underline{x} : \Sigma x \leq c \}$, where c is to be determined under H_0 so that the test is of size α i.e.

$$P[\underline{x} \in \mathscr{C} / H_0] = \alpha \Rightarrow P[\Sigma x \leq c / H_0] = \alpha \quad \text{—(1)}$$

let $Y = \Sigma X_i \sim G(nq, \theta)$, that is

$$f(y) = \frac{\theta^{nq} y^{nq-1}}{\Gamma nq} e^{-\theta y}, \quad y > 0$$

The constant 'c' depends on $\theta (= \theta_0)$ under H_0. So, the CR $\mathscr{C}$ (or constant 'c') is independent of θ_1 (under H_1) as long as $\theta_1 > \theta_0$. Hence the test based on this $\mathscr{C}$ is also UMP test of size α for $H_0: \theta = \theta_0$ Vs $H_1: \theta > \theta_0$

ii) The power function is

$$\pi(\theta) = P[\underline{x} \in \mathscr{C} / H_1] = P[Y \leq c \quad].$$

We know that $P[Y \leq c / H_0: \theta = \theta_0] = \alpha$

$$\Rightarrow \frac{\theta_0^{nq}}{\Gamma nq} \int_0^c y^{nq-1} e^{-\theta_0 y} dy = \alpha$$

Put $q = 1/n$

$$\Rightarrow \frac{\theta_0^{n(1/n)}}{\Gamma n \cdot 1/n} \int_0^c y^{n(\frac{1}{n})-1} e^{-\theta_0 y} dy = \alpha$$

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$$\Rightarrow \frac{\theta_0}{\Gamma 1} \int_0^c y^0 e^{-\theta_0 y} dy = \alpha$$

$$\Rightarrow \theta_0 \left[\frac{e^{-\theta_0 y}}{-\theta_0} \Big|_0^c \right] = \alpha$$

$$\Rightarrow -\left[e^{-\theta_0 c} - e^{-0} \right] = \alpha$$

$$\Rightarrow 1 - e^{-\theta_0 c} = \alpha$$

$$\Rightarrow e^{-\theta_0 c} = 1 - \alpha$$

$$\Rightarrow c = -[\ln(1-\alpha)]/\theta_0$$

Now $\pi(\theta) = P[Y \leq c] = P[Y \leq -\frac{1}{\theta_0} \ln(1-\alpha)]$

$$= P[Y \leq \log(1-\alpha)^{-1/\theta_0}]$$

$$= \int_0^{\log(1-\alpha)^{-1/\theta_0}} f(y)\,dy \quad \text{at } \xi = 1/n$$

$$= \int_0^{\log(1-\alpha)^{-1/\theta_0}} \frac{\theta^{n(\frac{1}{n})}}{\Gamma n(\frac{1}{n})} y^{n(\frac{1}{n})-1} e^{-\theta y} dy$$

$$= \theta \int_0^{\log(1-\alpha)^{-1/\theta_0}} e^{-\theta y} dy = \cancel{\theta} \frac{e^{-\theta y}}{-\cancel{\theta}} \Big|_0^{\log(1-\alpha)^{-1/\theta_0}}$$

$$= -\left[e^{-\theta \log(1-\alpha)^{-1/\theta_0}} - e^{-\theta(0)} \right]$$

$$= 1 - e^{-\theta \ln(1-\alpha)^{-1/\theta_0}} = 1 - \cancel{e}^{\cancel{\log}(1-\alpha)^{-\theta/-\theta_0}}$$

$$= 1 - (1-\alpha)^{\theta/\theta_0}. \text{ Hence proved.}$$

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Testing Composite Hypothesis

So far we have considered only simple H_0 i.e one which specifies the parameter values uniquely. There are, however, two important types of practical situations when H_0 is composite:

a) θ is a scalar parameter and we wish to test $H_0: \theta \leq \theta_0$ (or $\theta \geq \theta_0$) Vs $H_1: \theta > \theta_0$ (or $\theta < \theta_0$)

b) $\underline{\theta}$ is a vector parameter having 'p' components i.e. $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)$ (say) and we want to test a hypothesis about one or more of its components while the rest are ~~called Nuisance~~ unspecified (called nuisance parameters).

An exp: $X \sim N(\mu, \sigma^2)$, where $H_0: \mu = \mu_0$ Vs $H_1: \mu > \mu_0$ and σ^2 being unspecified. Both H_0 and H_1 are composite (as σ^2 is unknown) and σ^2 is called nuisance parameter. Because of composite nature of H_0, we have to revise our idea about the size of a test (i.e α).

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Since it is not possible to demand that $P[\underline{x} \in \omega / H_0 \text{ is true}] = \alpha$ for all values of θ in H_0 (for diffrnt values of θ, we get diffrnt $P(\underline{x} \in \omega / H_0)$). For each value of θ in H_0 this will change. Hence we amend our definition of size of a test. We say that a test of $H_0: \theta \in \Theta_0$ Vs $H_1: \theta \in \Theta_1$ is of size α if

$$P[\text{Reject } H_0 / H_0] \leq \alpha \quad \forall \theta \in \Theta_0 \text{ and}$$

$$P[\text{Reject } H_0 / H_0] = \alpha \text{ for at least one value of } \theta.$$

i.e α is the maximum probability of rejection of H_0 when H_0 is true.

We now examine the two statements given in (a) and (b).

<u>Single Paramter (Case a)</u>.

Suppose we want to test $H_0: \theta \leq \theta_0$ Vs $H_1: \theta > \theta_0$.

Then there will be a UMP test of size α in our amended sense.

In practical situations (for exp: Normal, exponential etc) the class of distn for which such test exist, is termed as the class with monotone likelihood ratio.

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Definition: Monotone Likelihood-Ratio

A family of densities $f_X(x;\theta)$, $\theta \in \Theta$, is said to have a monotone likelihood ratio if there exists a statistic, say $T = t(\underline{x})$ such that the ratio $\frac{f_X(\underline{x};\theta_2)}{f(\underline{x};\theta_1)}$ is either a non-decreasing function of $T(\underline{x})$ for every $\theta_1 < \theta_2$ or an non-increasing function of $T(\underline{x})$ for every $\theta_1 < \theta_2$.

Exp If $\{f(x;\theta), \theta \in \Theta\} = \{\frac{1}{\theta} I_{(0,\theta)}(x) : \theta > 0\}$, Then

$$\frac{L(\theta_2;\underline{x})}{L(\theta_1;\underline{x})} = \frac{\frac{1}{\theta_2^n}\prod_{i=1}^{n} I_{(0,\theta_2)}(x_i)}{\left(\frac{1}{\theta_1}\right)^n \prod_{i=1}^{n} I_{(0,\theta_1)}(x_i)} = \begin{cases} \left(\frac{\theta_1}{\theta_2}\right)^n \frac{I_{(0,\theta_2)}(y_n)}{I_{(0,\theta_1)}(y_n)}, & 0 < y_n < \theta_1 \\ 0, & \text{otherwise} \end{cases}$$

Now $\frac{L(\theta_2)}{L(\theta_1)}$ does not decrease for increasing value of Y_n. Therefore, the ratio is ~~non~~ non-decreasing or monotonically increasing function in $Y_n = \max(x_1, x_2, \ldots, x_n)$.

non-decreasing in $T(\underline{x})$ and if k is such (60)
that $P\{T(x) > k / H_0\} = \alpha$ then the test

(58)

Exp 1. let $X_1, X_2, \ldots, X_n$ be a r. sample from $B(1,p)$.
If we let $T = \Sigma X$, then the L.F is

$$f(\underline{x}, p) = p^{\Sigma x}(1-p)^{n-\Sigma x} = p^{t}(1-p)^{n-t}$$

For any $0 < P_1 < P_2 < 1$, the ratio

$$\frac{f(\underline{x}, p_2)}{f(\underline{x}, p_1)} = \frac{P_2^{T}(1-P_2)^{n-t}}{P_1^{t}(1-P_1)^{n-t}} = \left[\frac{P_2(1-P_1)}{P_1(1-P_2)}\right]^{t}\left(\frac{1-P_2}{1-P_1}\right)^{n} \quad (1)$$

is a function of $T = \Sigma X$ and the ratio (1) is
an increasing function of $T = \Sigma X$.

Thus, $B(1,p)$ has a monotone likelihood
ratio in the statistic $T = \Sigma X$.

Exp 2: let $X_1, \ldots, X_n$ be a r. sample from $N(\mu, \sigma^2)$
where σ^2 is known. The joint pdf $f(\underline{x}, \underline{\theta})$ is

$$f(\underline{x}, \mu) = (2\pi)^{-n/2}(\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\Sigma(x-\mu)^2}$$

Thus, for $\mu_1 < \mu_2$ (any two values of μ)
the ratio is

$$\frac{f(\underline{x}, \mu_2)}{f(\underline{x}, \mu_1)} = \frac{e^{-\frac{1}{2\sigma^2}\Sigma(x-\mu_2)^2}}{e^{-\frac{1}{2\sigma^2}\Sigma(x-\mu_1)^2}}$$

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$$\frac{f(\underline{x}, \mu_2)}{f(\underline{x}, \mu_1)} = e^{\frac{n(\mu_2-\mu_1)}{\sigma^2}\left[\bar{x} - \frac{1}{2}(\mu_1+\mu_2)\right]}$$

It can be seen from above ratio that this ratio depends on $\bar{x}$ and the ratio is an increasing function of $\bar{x}$ (for $\mu_1 < \mu_2$).

Thus, $N(\mu, \sigma_0^2)$ has monotone likelihood ratio in $T = \bar{x}$ (or Σx).

Theorem 1: Let $x_1, x_2, \ldots, x_n$ be a random sample from $f(x;\theta)$, where Θ is some interval. Assume that the family of densities $\{f(x;\theta), \theta \in \Theta\}$ has a monotone likelihood ratio in the statistic $T(\underline{x})$.

(i) If the monotone likelihood ratio is non-increasing in $T(\underline{x})$ and if k is such that $P[T(\underline{x}) < k / H_0] = \alpha$, then the test corresponding to the CR $\mathcal{C} = \{\underline{x} : T(\underline{x}) < k\}$ is UMP test of size α of $H_0: \theta \leq \theta_0$ Vs $H_1: \theta > \theta_0$.

ii) If the monotone likelihood ratio is

non-decreasing in $T(\underline{x})$ and if k is such that $P[T(\underline{x}) > k / H_0] = \alpha$, then the test corresponding to the CR $\mathcal{C} = \{\underline{x} : T(\underline{x}) > k\}$ is a uniformly most powerful test of size α of $H_0 : \theta \leq \theta_0$ Vs $H_1 : \theta > \theta_0$.

Exp (Non-regular) let $x_1, x_2, \ldots, x_n$ be a random sample from $f(x, \theta) = \frac{1}{\theta} I_{(0,\theta)}(x)$, $\theta > 0$.

Test $H_0 : \theta \leq \theta_0$ Vs $H_1 : \theta > \theta_0$.

In the previous example, we have shown that the family of densities has a monotone non-decreasing likelihood ratio in $T = Y_n$.

According to (ii) of the previous theorem, a UMP Test of size α is given by:

Reject H_0 if $Y_n = T(\underline{x}) > k$, where k is the solution to $P[Y_n > k / H_0] = \alpha$

$$\Rightarrow \alpha = \int_k^{\theta_0} f(y_n)\, dy_n \quad \text{——— } (*)$$

Note $f(y_n) = \frac{n}{\theta_0} \left(\frac{y_n}{\theta_0}\right)^{n-1}$, $0 < y_n < \theta$, under H_0.

$$\alpha = \int_k^{\theta_0} \frac{n}{\theta_0} \left(\frac{y_n}{\theta_0}\right)^{n-1} dy_n = \frac{n}{\theta_0^n} \left[y_n^n / n \Big|_k^{\theta_0} \right]$$

$$\alpha = \frac{1}{\theta_0^n} [\theta_0^n - k^n]$$

PTO

$$\alpha = 1 - \frac{k^n}{\theta_0^n} \Rightarrow k^n = \theta_0^n(1-\alpha)$$

$$\Rightarrow k = \theta_0(1-\alpha)^{1/n}.$$

~~Sover~~ Theorem 2:

Let $X_1, X_2, \ldots, X_n$ be a random sample from $f(x;\theta)$, where Θ is some interval. Assume that the family of densities $\{f(x;\theta), \theta \in \Theta\}$ belong to exponential family of distribution, that is

$$f(x;\theta) = a(\theta)\, b(x)\, e^{c(\theta) d(x)}$$

and let

$T(\underline{x}) = \Sigma d(\underline{x})$ (sufficient statistic)

(i) If $c(\theta)$ is a monotonic increasing function in θ and if there exists k such that

$P[T(\underline{x}) > k / H_0] = \alpha$, then the test with CR

$\mathcal{C} = \{\underline{x} : T(\underline{x}) > k\}$ is UMPT of size α of $H_0: \theta \le \theta_0$ Versus $H_1: \theta > \theta_0$ or of

$H_0: \theta = \theta_0$ Versus $H_1: \theta > \theta_0$.

ii) If $c(\theta)$ is a monotonic decreasing function in θ and if there exists k such that

$P[T(\underline{x}) < k/_{H_0}] = \alpha$, then the test with CR $\mathcal{C} = \{\underline{x} : T(\underline{x}) < k\}$ is UMP of size α for $H_0: \theta \le \theta_0$ Vs $H_1: \theta > \theta_0$ or for

$H_0: \theta = \theta_0$ Vs $H_1: \theta > \theta_0$.

Exp Consider $\underline{x}$ from $f(x,\theta) = \theta e^{-\theta x}, x > 0$.
$\Theta = \{\theta : \theta > 0\}$. Test $H_0 : \theta \leq \theta_0$ Vs $H_1 : \theta > \theta_0$.

Since $f(x;\theta) = \theta e^{-\theta x}$ belongs to exponential family with

$$f(x,\theta) = \theta \cdot 1 \cdot e^{(-\theta)(x)} = a(\theta) b(x) e^{c(\theta) d(x)}$$

$c(\theta) = -\theta$ is monotonic decreasing function of θ. $d(\underline{x}) = x$,

Thus, by Theorem 2, (result (ii) of Theorem 2)

The UMP Test is :

$\mathcal{C} = \{\underline{x} : \Sigma d(\underline{x}) = \Sigma x < k\}$ defines UMP Test of size α.

That is,

Reject H_0, iff $\Sigma x < k$, where k satisfies $P[\Sigma x < k / H_0] = \alpha$.

<u>Note</u>: Theorem 2 totally excludes the case of non-regular distributions as they are not members of exponential family.

Several Comments:

1) The null hypothesis was stated as $\theta \leq \theta_0$ in both theorems. If it had been stated as $\theta \geq \theta_0$, the two theorems would remain valid provided the inequalities that define the CRegions were reversed.

2) Theorem 1 is a consequence of theorem 2.

3) The theorems consider only one-sided hypotheses.

Unbiased Test:

There are many hypothesis testing problems for which a uniformly most powerful test does not exist. In these cases, it may be possible to restrict the class of tests and find a uniformly most powerful test in the restricted class. One such restricted class of tests that has some merit is the class of unbiased tests.

Definition: Consider a more general hypothesis of the type $H_0: \theta \in \Theta_0$ vs $H_1: \theta \in \Theta_1$. It is not possible to demand for

$$P[\underline{x} \in \mathcal{C} / H_1] \geq P[\underline{x} \in \mathcal{C} / H_0].$$

A test Ψ of the hypothesis
$H_0: \theta \in \Theta_0$ V/s $H_1: \theta \in \Theta_1$, is an U/b (unbiased) test iff

$$\sup_{\theta \in \Theta_0} \pi(\theta) \leq \inf_{\theta \in \Theta_1} \pi(\theta) \quad \text{——} (1)$$

that is when

$$\min P[\underline{x} \in k / H_0 : \theta \in \Theta_1] \geq \max P[\underline{x} \in k / H_1 : \theta \in \Theta_0]$$

Consequently in an U/b test the probability of rejecting H_0 when it is false is atleast as large as the probability of rejecting H_0 when it is true (i.e power of the test is atleast as large as size of the test).

If within this restricted class a test exists that is UMP, then it is UMPU (UMP unbiased) test. Theory has been developed by 'Lehman' for finding UMPU test.

Note: By class of U/b tests we mean that there may be many tests which are U/b (satisfying $\pi(\theta_1) \geq \pi(\theta_0)$ $\forall$ Θ_0, Θ_1). Now among these U/b tests, the test that gives maximum power as compared to all other U/b tests, is called UMPU test.
e.g $k_1, k_2, \ldots, k_k$ are all U/b tests. let $\pi_{k_1} \geq \pi(k_2, \ldots, k_k)$ $\forall$ $\theta \in \Theta$. Then k_1 gives UMPU Test (in the restricted class).

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